April 3rd (6:00 am) Key Features of Quantum Mechanics.

- Energy of photon is $\underbrace{E}_{J} = \underbrace{h}_{J\cdot s}\underbrace{\nu}_{s^{-1}}$, where ν is color. Think of THAT as a Brandon Sanderson twist.

So the units of $\hbar = h/2\pi$ are energy · second = action.

integral of <u>energy</u> along a path, where it picks up <u>time</u>

- Light linearly polarized in the $\hat{x}$ direction means the electric field lies along the $\hat{x}$ axis while oscillating back and forth.

- $i\hbar \frac{\partial \Psi}{\partial t} - \hat{H}\Psi = 0 \Rightarrow \hat{H}$ has units of energy since $\hbar$ is $J \cdot s$ and $\frac{d}{dt}$ has units of s^{-1}.

- Arbitrary superposition of two states:

$$|\psi\rangle = \cos\frac{\theta}{2}|A\rangle + \sin\frac{\theta}{2}\, e^{i\phi}|B\rangle, \quad \theta \in (0,\pi),\ \phi \in [0,2\pi)$$

comes from

$$|\psi\rangle = \alpha|A\rangle + \beta|B\rangle, \quad \alpha,\beta \in \mathbb{C}$$

$$\Rightarrow |\psi\rangle = |A\rangle + \frac{\beta}{\alpha}|B\rangle, \quad \alpha,\beta \in \mathbb{C}$$

$$\Rightarrow |\psi\rangle = |A\rangle + re^{i\theta}|B\rangle,$$

$$\Rightarrow |\psi\rangle = |A\rangle + \tan(\nu)\, e^{i\theta}|B\rangle, \quad \nu \in [0,\tfrac{\pi}{2}),\ \theta \in [0,2\pi).$$

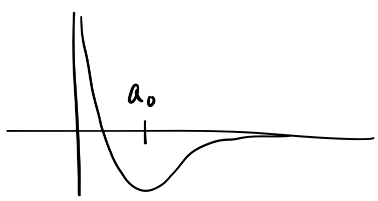
Hence directionality in 3D space.

- Spin is angular momentum. Measured as $\pm\frac{\hbar}{2}$, the units are of $\hbar = J\cdot s = kg\,\frac{m^2}{s}$. Makes sense from $\vec{L} = \vec{r}\times\vec{p} = m\cdot kg\,\frac{m}{s}$.

- Proton has charge e, electron has charge $-e$. Coulomb potential of proton-electron system is $\frac{q_1 q_2}{r} = \frac{-e^2}{r}$.

- Units match up in the uncertainty relation $\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2}$. Similarly in $[\hat{x},\hat{p}] = i\hbar$.

- Stability of atoms arises from uncertainty: electron localized too close to nucleus picks up huge Δp and thus kinetic energy.



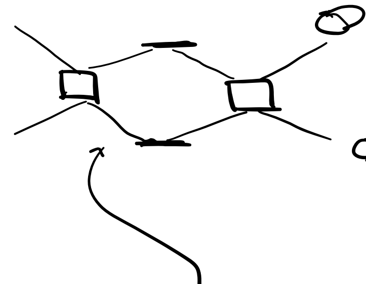
Since $\Delta p \sim \hbar/\Delta x \sim \hbar/r$ and $K \sim \frac{(\Delta p)^2}{2m} \sim \frac{\hbar^2}{2mr^2}$ gives total energy $E \sim \frac{\hbar^2}{2mr^2} - \frac{e^2}{r}$, the minimum is when $\frac{-\hbar^2}{mr^3} + \frac{e^2}{r^2} = 0 \Rightarrow r = \frac{\hbar^2}{me^2}$, called $\underline{a_0}$.

- Useful numbers to memorize:

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}, \quad m_e c^2 \approx 0.511 \text{ MeV}, \quad \hbar c \approx 197.3 \text{ MeV}\cdot\text{fm}.$$

10^{-15} m

April 4th (6:00 am) Light, Particles, and Waves



- Mach-Zehnder interferometers demonstrate basic unitaries and state change.
- Elitzur-Vaidman bomb detection by placing bomb here to see if it blocks interference pattern or not.
- Photoelectric effect: photon with energy $E = h\nu$ greater than W, the "work function" of the polished metal, ejects electron with kinetic energy $h\nu - W$.
- Surprise! $[h]$ has units of angular momentum.
- Canonical length to any particle of mass m is Compton wavelength $\lambda_c = \frac{h}{mc}$ by forming a momentum mc.
- Lots of practice with $\frac{e^2}{\hbar c} \approx \frac{1}{137}$, $m_e c^2 \approx 0.511$ MeV, $\hbar c \approx 197.3$ MeV·fm
- DeBroglie wavelength uses true momentum: $\lambda = \frac{h}{|p|}$.
- $\lambda = \frac{h}{p} \Rightarrow p = \frac{h}{\lambda} = \hbar \frac{2\pi}{\lambda} = \hbar k$, with $k \equiv \frac{2\pi}{\lambda}$ the wave number. (units of m^{-1})
- DeBroglie relations:

angular frequency $= 2\pi\nu$, hence $\hbar$!

$$p = \hbar k, \quad E = \hbar\omega.$$

April 5th (6:00 am) Schrödinger Equation

- Free particle wavefunction $\Psi(x,t) = e^{i(kx-\omega t)}$ has $E = \hbar\omega$, $p = \hbar k$.
- $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ exactly extracts $\hbar k$! Units: $[\hbar \frac{\partial}{\partial x}] = [p]$, and $\frac{1}{i}$ so it's real.
- $\hat{E} = \frac{\hat{p}^2}{2m}$ and also $\hat{E} = i\hbar \frac{d}{dt}$ gives free particle Schrödinger:

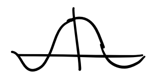
$$\left[i\hbar \frac{d}{dt}\right]\Psi = \left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\right]\Psi .$$

- With energy $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x},t)$, more generally

$$i\hbar \frac{d}{dt}\Psi = \hat{H}\Psi .$$

- Define $\hat{x} f(x) = x f(x)$ — not eigenstate unless $f(x) \sim \delta(x_0)$.
- $[\hat{x},\hat{p}] = i\hbar$ because $\frac{\hbar}{i}\left(x\frac{\partial}{\partial x} - \frac{\partial}{\partial x}x\right) = \frac{\hbar}{i}\left(\cancel{x\frac{\partial}{\partial x}} - \left(\mathrm{Id} + \cancel{x\frac{\partial}{\partial x}}\right)\right) = \frac{-\hbar}{i} = i\hbar$.
- Paulis! $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
  Commutators are $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. Always normalize eigenvectors!
- Postulate $\int d^n x\, |\Psi(x,t)|^2 = 1$. Thus units of Ψ are $L^{-n/2}$.
- In 1D, this requires $\lim_{x\to\pm\infty} \Psi(x,t) = 0$, $\lim_{x\to\pm\infty} \left|\frac{\partial\Psi}{\partial x}\right| < \infty$.
- Conservation of probability forces $\hat{H}^\dagger = \hat{H}$, Hermiticity.
- Probability current $J(x,t) = \frac{\hbar}{m}\mathrm{Im}\left(\Psi^* \frac{\partial\Psi}{\partial x}\right)$ satisfies $\frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = 0$.
  Units are $[J] = s^{-1}m^{-(n-1)}$ in n spatial dimensions, e.g. $s^{-1}m^{-2}$ for 3D.
- Useful integral: $\int_0^\infty dx\, x^n e^{-x} = n!$
- Useful trick: when asked to approximate, APPROXIMATE!

April 6th (6:00 am) Wave Packets, Uncertainty, and Momentum Space

- Wave packet is general superposition $\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \overbrace{\Phi(k)}^{\text{arbitrary function of } k}\, \underbrace{e^{ikx}}_{\text{plane wave}}$
- Inverse Fourier transform: $\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \Psi(x,0)\, e^{-ikx}$.
- Key intuition: $k|x|$ too large means the phase e^{ikx} washes out contributions from $\Phi(k)$. Thus $\Delta k \Delta x \approx \text{const}$ (loosely).
- Another intuition: step $\underset{\Delta k}{\sqcap}\ \sqrt{\Delta k}$ contributes $\sim \sqrt{\Delta k}\, \frac{\sin(\Delta k\, x)}{\Delta k\, x}$ 
- Wave packet changes shape: $\overbrace{\frac{\Delta p}{m}}^{\text{velocity uncertainty}} t \ll \underbrace{\Delta x}_{\text{relevant lengthscale}}$ to remain localized.
- Time evolution of wavepackets: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \Phi(k) e^{ikx} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \Phi(k)\, e^{i(kx - \underbrace{\frac{\hbar k^2}{2m} t}_{\text{energy eigenstate evolution}})}$.
- Energy of plane wave $e^{i(kx-\omega t)}$ is $\frac{\hbar^2 k^2}{2m}$.
- Define $\delta(x-x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x_0)}$. Then $\delta(ax) = \frac{1}{|a|} \delta(x)$.
- Plancherel's Theorem: $\int dx\, |\Psi(x)|^2 = \int dk\, |\Psi(k)|^2$.
- Momentum space is born with $\tilde{\Phi}(p) \equiv \Phi(\frac{p}{\hbar})$, $|\tilde{\Phi}(p)|^2$ a probability.
- Postulate $\hat{p}\, \Phi(p) = p\, \Phi(p)$ (of functions, not eigenvalue), and $\hat{x} = i\hbar \frac{\partial}{\partial p}$.

April 6th (1:00pm) Expectation Values

- Define $\langle \hat{Q} \rangle_{|\psi\rangle} = \int dx\, \psi^*(x,t)\, \hat{Q}\, \psi(x,t)$.
- $\langle L_i \rangle = 0$ in a rotationally invariant state.
- Time dependence $i\hbar \frac{d\langle \hat{Q} \rangle}{dt} = \langle [\hat{Q}, \hat{H}] \rangle$ for $\frac{d\hat{Q}}{dt} = 0$.
- Helpful formula: $[\hat{p}, f(\hat{x})] = \frac{\hbar}{i} \frac{\partial f(\hat{x})}{\partial \hat{x}}$.
- Define inner product, Hermitian, and state the Spectral Theorem.
- Measurement axiom: collapse into $|\psi_i\rangle$ with probability $|\alpha_i|^2$ where the original wavefunction is $|\psi\rangle = \sum_{i=1}^{\infty} \alpha_i |\psi_i\rangle$ for $\hat{Q}|\psi_i\rangle = \alpha_i |\psi_i\rangle$ and Hermitian $\hat{Q}$.
- Particle on a circle: $\psi(x+L) = \psi(x)$ gives $kL = 2\pi n$, $n \in \mathbb{Z}$, for $\hat{p}$ eigenstates.
- Uncertainty $(\Delta \hat{Q})^2 = \langle \hat{Q}^2 \rangle - \langle \hat{Q} \rangle = \langle (\hat{Q} - \langle \hat{Q} \rangle)^2 \rangle = \| (\hat{Q} - \langle \hat{Q} \rangle) |\psi\rangle \|^2$. The last characterization shows $(\Delta \hat{Q})^2 = 0 \iff \hat{Q}|\psi\rangle = \langle \hat{Q} \rangle |\psi\rangle$, i.e. eigenstates have zero uncertainty.

April 7th ($6{:}00^{am}$) Stationary States: Special Potentials

- Stationary states are separable as $\Psi(x,t) = g(t)\psi(x)$. The solution is $\Psi(x,t) = e^{-iEt/\hbar}\psi(x)$ with $\hat{H}\psi = E\psi$. Notice the units in the exponential cancel as they should.
- $\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ with ψ_1, ψ_2 having energies E_1, E_2 will eventually evolve into $\frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$ since the phases grow differently.
- Intuition: whatever features $V(x)$ contains leak into $\psi''(x)$. This is straight out of the Schrödinger equation.

  $V(x)$ continuous $\to$ $\psi''(x)$ continuous

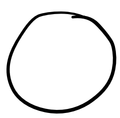
  $V(x)$ finite discontinuities $\to$ $\psi''(x)$ finite discontinuities

  $V(x)$ δ functions $\to$ $\psi''(x)$ δ functions $\to$ $\psi'(x) = \int \psi''(x)$ has finite discontinuities

  $V(x)$ hard wall $\to$ $\psi''(x)$ is "infinite" to compensate

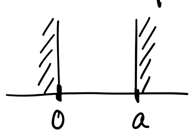
- Classic examples:

  Particle on a circle: $\psi_n(x) = e^{ik_n x}$, $E = \frac{\hbar^2 k_n^2}{2m}$, $k_n = \frac{2\pi}{L}n$, $n \in \mathbb{Z}$,

  $\hat{x}$ ill-defined due to the new topology.

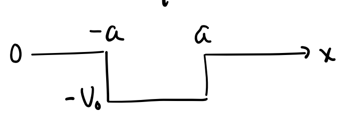
  Infinite Square Well: $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$, $n \in \mathbb{N}^+$.

  A node is a point where $\psi(x)$ changes sign, i.e. $\psi(x_0) = 0$ but $x_0 \notin \{0, a\}$.

  $\hat{p}$ ill-defined now due to boundary conditions ($\hat{p}^2$ okay!)

  Finite Square Well: $\psi''(x) = \frac{-2m}{\hbar^2}(E - V(x))\psi(x) \to \cos(kx)$ in classically allowed,

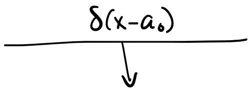
  

  $\exp(-\kappa x)$ in classically forbidden region, decaying so normalizable, and stitched together. Beautiful redefinition as $\eta = ka$, $\xi = \kappa a$, $z_0 = \frac{2mV_0 a^2}{\hbar^2}$ so solutions to even bound states are

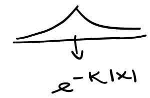
  $$\eta^2 + \xi^2 = z_0^2, \quad \xi = \eta \tan\eta, \quad \xi, \eta > 0.$$

Odd bound states have $\xi = -\eta \cot \eta$.

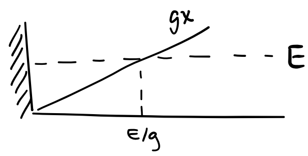
Delta Potential



Integrate $\Psi(x)$ across $\delta(x-x_0)$ from $-\varepsilon$ to ε to capture the discontinuous jump of Ψ' at x_0 .



Linear Potential

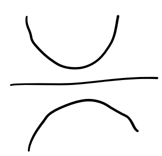
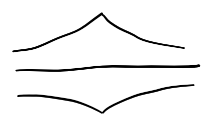


Solve in momentum space where $\left[\frac{p^2}{2m} + V\left(i\hbar \frac{d}{dp}\right)\right]\phi(p) = E\,\phi(p)$, then Fourier transform into position space. Airy functions!

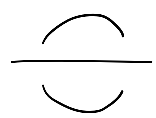
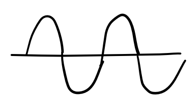
April 7th (3:00pm) Stationary States: General Features

- $\nexists$ eigenstates with energy $E < \inf(V(x))$.
- $\nexists$ degenerate <u>bound</u> states in 1D potential.
- We can always work with a real wavefunction if $V(x)$ is real.
- If $V(x) = V(-x)$, then $[\hat{P}, \hat{H}] = 0$ so we can choose even/odd energy eigenstates. Moreover, bound states must be even or odd by lack of degeneracy.
- Semiclassical approximation sets $\lambda(x) = \frac{h}{p(x)}$, requires $\left|\frac{d\lambda}{dx}\right| \ll 1$, says $\psi(x) \sim \frac{1}{\sqrt{p(x)}}$. Intuition: particle is more likely to be where it classically spends more time, i.e. where its velocity is slow.
- Sketching wavefunctions: $\frac{\psi''(x)}{\psi(x)} = -\frac{2m}{\hbar^2}(E - V(x))$

  Classically forbidden $E < V(x)$: convex toward the axis

  like  or .

  Classically allowed $E > V(x)$: concave toward the axis

  like  or .

  Turning point $E = V(x)$: inflection point where $\psi''(x)$ changes sign.

- Both $\psi(x)$ and $\psi'(x)$ cannot simultaneously vanish because the Schrödinger equation is second order, so these would imply $\psi(x) \equiv 0$.
- Intuition for quantization of even bound state energy: we need $\psi(0) = 0$ (odd) or $\psi'(0) = 0$. Any old E numerically extended via the Schrödinger equation into a wavefunction might fail this condition! Now it's clear why the n^{th} excited state has n nodes. (Visualize increasing E. Each successful E adds another node.)

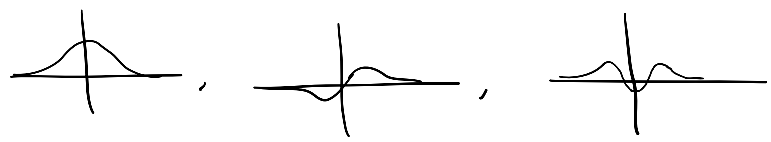
April 8^{th} ($6:00^{am}$) Stationary States: General Features

- To make it formal, the Node Theorem states there are n nodes for the n^{th} excited state.
- Shooting method to numerically solve for energies: ~~start with~~ $\psi(0)=0, \psi'(0)=1$ (odd), or $\psi(0)=1$, $\psi'(0)=0$ (even), then binary search E values that don't make $\psi(x)$ blow up as you integrate forward using the Schrödinger equation. Works for even potentials only!
- Remove units from the Schrödinger equation by constructing characteristic length and energy scales, L and E_0, from $\hbar$, m, and parameters in V(x). Useful to do so for numerical integration. Then substitute $x = Lu$, $\varepsilon = E/E_0$ and rewrite Schrödinger.
- If $\hat{Q}$ and $\hat{H}$ are time independent and $\hat{H}|\psi\rangle = E|\psi\rangle$, then $\langle[\hat{Q},\hat{H}]\rangle_{|\psi\rangle} = 0$. (Let $\hat{H}$ act both directions.)
- Virial Theorem pops out when $\hat{Q} = \hat{x}\hat{p}$. Then $\left\langle \frac{\hat{p}^2}{2m} \right\rangle = \frac{1}{2}\left\langle \hat{x}\frac{dV}{d\hat{x}} \right\rangle$.
- Variational principle: $E_{gs} \le \langle\psi|\hat{H}|\psi\rangle$ for any $\|\psi\| = 1$ wavefunction. Even better, parametrize $\psi(\beta_1,\dots,\beta_n)$ and minimize over $(\beta_1,\dots,\beta_n)$. The wavefunction $|\psi\rangle$ doesn't need to be an energy eigenstate.
- Feynman-Hellman Lemma states $\frac{dE_n(\lambda)}{d\lambda} = \left\langle \frac{d\hat{H}}{d\lambda} \right\rangle_{|\psi_n\rangle}$ for λ a parameter of the Hamiltonian. (Proof is short and pretty! Page 198.) Only works for nondegenerate eigenstates. When $\hat{H} = H^{(0)} + \lambda\delta H$, this result reproduces $E_n^{(1)} = \langle\psi_n|\delta H|\psi_n\rangle$.

April 9th (6:00 am) Stationary States: Scattering

- Scattering state means nonnormalizable.
- Finite step potential : postulate $\Psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Ce^{-ik'x} & x \geq 0 \end{cases}$ for wave moving right (A transmitted, B reflected) and another left. Solve for B, C in terms of A by requiring Ψ, Ψ' continuous at 0.
- For $E < V_0$, $|A| = |B|$ and $B = -Ae^{2i\delta(E)}$, where $\delta(E)$ is the phase shift of the reflected wave and is $\delta(E) = \tan^{-1}\sqrt{\frac{E}{V_0 - E}} \in [0, 2\pi]$.
- Intuition: the physical (normalizable) scenario is always a packet of scattering states that could represent a real particle.
- For $E < V_0$, reflected wave experiences a time delay $\frac{1}{2}\sqrt{\frac{1}{E(V_0 - E)}}$.
- Resonant transmission across a finite square well barrier means $T = 1$ for energies that are also energies of the infinite square well. This is crazy! Some waves ignore the potential!
- Ramsauer-Townsend effect is shooting electrons through noble gas atoms. As electron energies increased, scattering went from high to zero to high to zero to high! This is resonant transmission in a spherical well.

April 10th (6:00am) Harmonic Oscillator

- Characteristic energy $\hbar\omega$, length scale $\sqrt{\frac{\hbar}{m\omega}}$ (check the units!)
- $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \longrightarrow \frac{d^2\varphi}{du^2} = (u^2 - \varepsilon)\varphi$ unit-free differential equation
- Guess solution is of the form $\varphi(u) = h(u)e^{-u^2/2}$ for $h(u)$ a polynomial. New differential equation to satisfy is $h'' - 2uh' + (\varepsilon - 1)h = 0$. Solutions of degree j are quantized, $\varepsilon = 2j+1$. Series for h must terminate for ε to make a normalizable state (done by constructing a recurrence relation). These are Hermite polynomials.
- Key physical result: $E = \hbar\omega(n + \frac{1}{2})$ for $n = 0, 1, 2, \ldots$
- First few Hermite polynomials are $1, 2u, 4u^2 - 2, \ldots$
- Eigenstates $h(u)e^{-u^2/2}$ are 
- Algebraically, factorize $\frac{1}{2}m\omega^2\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right) = \frac{1}{2}m\omega^2\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$. Failure to commute gives additive constant. Remove units by defining $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$, $\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$. Then $\hat{H} = \hbar\omega(a^+a + \frac{1}{2})$.
- Key intuition: $\hat{a}, \hat{a}^+$ are defined the only way possible consistent with units and $\hat{a} \sim \hat{x} + i\hat{p}$, $\hat{a}^+ \sim \hat{x} - i\hat{p}$, $[\hat{a}, \hat{a}^+] = 1$.
- Inverses are key as well: $\hat{x} \sim \hat{a} + \hat{a}^+$, $\hat{p} \sim \hat{a} - \hat{a}^+$.
- For quick commutator tricks, identify $\hat{a} = \frac{d}{d\hat{a}^+}$, $\hat{a}^+ = -\frac{d}{d\hat{a}}$. These are just exactly what makes $[\hat{a}, \hat{a}^+] = 1$ consistent.
- Commutator trick: if $A|\psi\rangle = 0$, then $AB|\psi\rangle = [A,B]|\psi\rangle$. Useful to show $\langle 1|1\rangle = \langle 0|aa^+|0\rangle = \langle 0|[a,a^+]|0\rangle = \langle 0|0\rangle = 1$.
- Creation and annihilation: $\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$.

April 11th (6:00am)

Angular Momentum and Central Potentials

- Classical origin of orbital/spin angular momentum is $J = R \times mV + \sum_i r_i \times (mv_i)$, with $R = \sum r_i$, $V = \sum \dot{r}_i$ the center of mass versions. So $J = L + S$ = orbital + spin.
- Intuition: think of translation operator $U_a = \exp(-ia\hat{p}/\hbar)$ as creating a shifted Taylor expansion $1 - a\frac{d}{dx} + \frac{a^2}{2}\frac{d^2}{dx^2} - \dots$.
- Momentum generates translations: $x \to x+a$ by $e^{-ia\hat{p}/\hbar}$. Angular momentum generates rotations: $\phi \to \phi + \alpha$ by $e^{-i\alpha\hat{L}_z/\hbar}$.
- Define $\hat{L} = \hat{r} \times \hat{p}$ (good to remember units!), e.g. $\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$.
- Classic commutators $[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$, $[L^2, L_i] = 0$.
- In spherical, $\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$ $\left(\text{because } \frac{\partial}{\partial\phi} = \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right.$ since $\frac{\partial}{\partial\phi}(R\sin\theta\sin\phi) = R\sin\theta\cos\phi = \hat{x}$ for example$\left.\right)$.
- Intuition: $\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$ gives it the same topology as $\hat{p}$ on a circle with $x_0 \sim x_0 + 2\pi$. This gives rise to quantization of m.
- $\hat{L}_z|\psi\rangle = \hbar m|\psi\rangle$, $m \in \mathbb{Z}$, arises thus.

  $\hat{L}^2|\psi\rangle = \hbar^2 \ell(\ell+1)|\psi\rangle$, $\ell \in \mathbb{N}$, arises by recognizing the differential equation for the spherical harmonics as a Legendre diff. eq.

  Then $-\ell \le m \le \ell$ arises from ensuring $P_\ell^m(x) = \dots \left(\frac{d}{dx}\right)^{|m|} P_\ell(x) \neq 0$.
- Intuition for solid angles: $\int d\Omega \equiv \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)$. (Equals 4π.)
- Spherical harmonics fall out to give radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2 u_{E\ell}}{dr^2} + \left(V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right) u_{E\ell} = E u_{E\ell}$$

with $u_{E\ell}(r) = rR_{E\ell}(r)$ the radial component. New "centrifugal" term.

- **Intuition**: the reason we need $r^2 dr\, d\Omega$ in integrals (and later for density of states) is both Jacobian and $r^2 d\Omega$ being geometrically the "area-infinitesimal" for a radius r sphere.
- As $r \to 0$, $u \sim r^{\ell+1}$ (since $\frac{d^2u}{dr^2} = \frac{\ell(\ell+1)}{r^2} u \Rightarrow u \sim r^s \Rightarrow s = \ell+1$ or $-\ell$).
- Intuition: just remember $\psi(0) \neq 0$ only for $\ell = 0$.

  Then $\psi \sim r^n$ as $r \to 0$ forces $n = \ell$.

April 12th (6:00am) Hydrogen Atom

- Wavefunction of two particles is $\Psi(\vec{x}_a, \vec{x}_b)$ with $\int |\Psi(\vec{x}_a,\vec{x}_b)|^2 d^3x\, d^3x = 1$. This is a joint probability density, not two wavefunctions! We use center-of-mass frame to reduce to one relative position vector.
- Recall our scales: $a_0 = \frac{\hbar^2}{me^2} \approx 0.53$ Å, $\alpha = \frac{e^2}{\hbar c} \approx 1/137$, $Ryd = \frac{e^2}{2a_0} = \alpha^2 \frac{1}{2} mc^2 \approx 13.6$ eV.
- Solving the radial equation with series leads to quantization. It's not that interesting though. Key point is (n, ℓ, m) with $E_n = \frac{Z^2 e^2}{2a_0} \frac{1}{n^2}$, $n > 0$, $0 \leq \ell < n$, and $-\ell \leq m \leq \ell$. To center your intuition, remember $n=1$ has one state (i.e. $\ell = 0$). And $n=1$ is the ground state not $n=0$ so that $E_n \sim \frac{1}{n^2}$ works out.
- Some patterns: $\Psi_{n\ell m}(r) \sim (\cdots) e^{-r/na_0}$ lets you read off n. Fix ℓ and $n = \ell+1$ is the ground state. Thus when $n = \ell + 1 + N$, the Node Theorem says there are $N = n - \ell - 1$ nodes.
- $\langle r \rangle = \frac{1}{2} a_0 (3n^2 - \ell(\ell+1))$, $\langle \frac{1}{r} \rangle = \frac{1}{a_0 n^2}$, $\langle \frac{1}{r^2} \rangle = \frac{1}{a_0^2 n^3 \ell(\ell + \frac{1}{2}}$.

April 13th (6:00am) Spin 1/2

- A vector in a 2D complex vector space is known as a spinor.
- $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $S_i = \frac{\hbar}{2}\sigma_i$, $[S_i, S_j] = i\hbar\varepsilon_{ijk} S_k$.
- Magnetic dipole $\vec{\mu} = \frac{I}{c}\vec{A}$ is classically current through a loop. Energy is $E = -\vec{\mu}\cdot\vec{B}$: dipole tends to align with $\vec{B}$. Units are $[\vec{\mu}] = \frac{\text{erg}}{\text{gauss}} = \text{Coul}\cdot\text{s}$, $[B] = \text{gauss}$, erg ~ energy.

- For an electron, $\hat{\mu} = -g\,\frac{e\hbar}{2m_e c}\,\frac{\hat{S}}{\hbar}$, $g = 2$.

Define the Bohr magneton $\mu_B = \frac{e\hbar}{2m_e c} \approx 5.788\times10^{-9}\,\frac{\text{eV}}{\text{gauss}}$.

- Spin in arbitrary direction is $S_{\vec{n}} = \frac{\hbar}{2}\,\vec{n}\cdot\vec{\sigma}$,

$|n;+\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$, $|n;-\rangle$ by orthogonality.

April 14th (6:00 am) Vector Spaces and Operators

- Examples are many quantum systems as vector spaces.
- Explains ε_{ijk} and our favorite $\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$.
- Paulis anticommute: $\sigma_1\sigma_2 = -\sigma_2\sigma_1$.
- Raising operator $S_\pm = S_x \pm i S_y = \begin{pmatrix} 0 & 1\pm 1 \\ 1\mp 1 & 0 \end{pmatrix}$ gives raising intuition.
- $e^{iM\theta} = \cos\theta \, \mathbb{1} + iM\sin\theta$, if $M^2 = \mathbb{1}$.
- Hadamard's lemma: $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$
- Baker-Campbell-Hausdorff: $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ if $[A,[A,B]] = [B,[A,B]] = 0$.

April 15th (6:00am) — Inner Product, Adjoints, and Bra-kets

- Cauchy-Schwarz: $|\langle a|b\rangle|^2 \leq \|a\|^2 \|b\|^2$.
- $P^2 = P$ and $P^\dagger = P$ means orthogonal projector.
- Unitary means surjective and $\|Uu\| = \|u\| \ \forall u$, or $U^\dagger U = UU^\dagger = \mathbb{1}$.
- If M is Hermitian then e^{iM} is unitary. Unitaries send orthonormal bases to orthonormal bases.
- Rotation operator $R_{\vec{n}}(\alpha) = \exp(-i\alpha S_{\vec{n}}/\hbar)$. This works in any angular momentum algebra.
- Resolution of the identity $\mathbb{1} = \sum_k |k\rangle\langle k|$ is surprisingly useful in computations.

April 16th (7:00am) Uncertainty Principle and Compatible Operators

- $(\Delta\hat{Q})^2 = \langle\hat{Q}^2\rangle - \langle\hat{Q}\rangle^2$ is like an orthogonal projection, as seen from $\|\hat{Q}\Psi\|^2 = \langle\hat{Q}\rangle^2 + (\Delta\hat{Q})^2$. Uncertainty $(\Delta\hat{Q})^2$ has units $[\hat{Q}^2]$.

- $(\Delta A)^2(\Delta B)^2 \geq |\langle \frac{1}{2i}[A,B]\rangle|^2$
- Together with $\frac{\hbar}{i}\frac{d\langle\hat{Q}\rangle}{dt} = \langle[\hat{H},\hat{Q}]\rangle$, energy time uncertainty with $(\Delta t_Q)^2 = \frac{(\Delta Q)^2}{|\frac{d\langle Q\rangle}{dt}|^2}$ is $(\Delta H)^2(\Delta t_Q)^2 \geq \frac{\hbar}{2}$.

- A year is about $\pi \times 10^7$ seconds (accurate to 1% !)
- Lower bounds with the uncertainty principle: in $\langle\hat{H}\rangle$, relax $\langle p^2\rangle$ to $(\Delta p)^2$ and $\langle x^4\rangle$ to $(\Delta x)^4$, then relate $(\Delta p)^2 \geq \frac{\hbar}{2(\Delta x)^2}$ and minimize $\langle\hat{H}\rangle$ as a function of $(\Delta x)^2$.

- If $T|\Psi\rangle = \lambda|\Psi\rangle$ then $T^\dagger|\Psi\rangle = \lambda^*|\Psi\rangle$.
- Normal operators $([A,A^\dagger])$ are $A = \sum_{k=0}^{\infty}\lambda_k P_k$, with P_k a complete set of orthogonal projectors $P_k^2 = P_k$, $P_k P_\ell = \delta_{k\ell} P_\ell$, $\sum_{k=0}^{\infty} P_k = \mathbb{1}$.
- Complete set of commuting observables (CSCO) resolves all degeneracies. CRITICAL to determine the commuting observables in a given physical problem.

April 17th (6:00am) Pictures of Quantum Mechanics

- Unitary time evolution: postulate $|\Psi(t)\rangle = \mathcal{U}(t,t_0)|\Psi(t_0)\rangle \; \forall t, t_0$. It implies the Schrödinger equation by differentiating the above.
- Three cases of time evolution:

  $\hat{H}$ time-independent: $\mathcal{U}(t) = \exp(\hat{H}t/i\hbar)$

  $[\hat{H}(t_0), \hat{H}(t_1)] = 0 \; \forall t_0, t_1$: $\mathcal{U}(t) = \exp\left(\frac{1}{i\hbar}\int_0^t \hat{H}(t')\,dt'\right)$

  General: $\mathcal{U}(t) = \mathbb{1} + \frac{1}{i\hbar}\int_0^t dt_0'\,\hat{H}(t_0') + \frac{1}{(i\hbar)^2}\int_0^t dt_0'\,\hat{H}(t_0)\int_{t_0}^t dt_1'\,\hat{H}(t_1) + \cdots$
- Heisenberg picture moves time dependence into the operators: $A_H(t) = \mathcal{U}^\dagger(t)\hat{A}_S\,\mathcal{U}(t)$, $\langle \hat{A}_S\rangle_{|\Psi(t)\rangle} = \langle \hat{A}_H(t)\rangle_{|\Psi(0)\rangle}$.
- Evolution is $i\hbar\frac{dA_H}{dt} = [A_H, H_H] + i\hbar\frac{dA_H}{dt}$.
- Find Heisenberg operators for the harmonic oscillator by setting up a 2nd order ODE for $\hat{x}_H(t)$ using eq. of motion.

April 18^{th} ($6{:}00^{am}$) Dynamics of Quantum Systems

- Coherent state $e^{-i\hat{p}x_0/\hbar}|0\rangle$ oscillates classically and maintains minimum uncertainty shape. To compute with it, remember the Heisenberg picture $\hat{x}_H$, $\hat{p}_H$.
- Using $e^{A+B} = e^A e^B e^{\frac{1}{2}[A,B]}$ and expanding $\hat{p}$ as $\frac{im\omega L_0}{\sqrt{2}}(\hat{a}-\hat{a}^\dagger)$, coherent states are a Poisson distribution of energy eigenstates.
- Generally, $\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle$ rotates $\alpha \to \alpha - \omega t$ in time.

  Physically, $\alpha = \frac{1}{\sqrt{2}}\left(\frac{\langle\hat{x}\rangle}{L_0} + i\frac{L_0\langle\hat{p}\rangle}{\hbar}\right)$. So beautiful!

  For more intuition, $\alpha\hat{a}^\dagger - \alpha^*\hat{a} = \frac{i}{\hbar}(\hat{p}\langle\hat{x}\rangle - \langle\hat{p}\rangle\hat{x})$: double translation!

- E&M field operators from $E \sim \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2)$ giving a harmonic oscillator. Strategy: find conjugate operators s.t. $[\hat{p},\hat{q}] = i\hbar$, then define raising/lowering $\hat{a}, \hat{a}^\dagger$ to re-create $\hat{H} = \hbar\omega(\hat{a}^\dagger a + \frac{1}{2})$.
- Taste of quantum field theory: coherent state of photons $|\alpha\rangle$ contains about $\langle\hat{N}\rangle = |\alpha|^2$ photons as a standing wave.
- Reminder: $\mu_B = \frac{e\hbar}{2m_e c}$, $E = -2\mu_B \frac{\hat{S}}{\hbar}\cdot\vec{B}$ (check units!).
- Fun fact: neutrons have a magnetic dipole moment!
- Nuclear magnetic resonance: spin precession with frequency $\omega_L = -\gamma B$, $\hat{\mu} = \gamma\hat{S}$. Rotating magnetic field

$B_0\hat{z} + B_1(\hat{x}\cos\omega t - \hat{y}\sin\omega t)$ gives $|\psi, t\rangle = \exp\left(\frac{i\omega t S_z}{\hbar}\right)\exp\left(i\frac{\gamma B_R \cdot \hat{S}\, t}{\hbar}\right)|\psi, 0\rangle$,

$B_R = B_1\hat{x} + B_0\left(1 - \frac{\omega}{\omega_0}\right)\hat{z}$, $\omega_0 \equiv \gamma B_0$, $\hat{x}, \hat{y}, \hat{z}$ vectors not operators.

- Arbitrary 2x2 systems can be considered spin precession.
- Many algebraic results (and "supersymmetry") fall out of general factorized Hamiltonians $A^\dagger A$, $AA^\dagger$.

April 19th (6:00 am)

Multiparticle States and Tensor Products

- $S \otimes T\,(u \otimes v) = Su \otimes Tv$ is the fundamental rule. Inner products are $\langle u \otimes v, \tilde{u} \otimes \tilde{v} \rangle = \langle u, \tilde{u} \rangle \langle v, \tilde{v} \rangle$.
- $A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}$, e.g. $\sigma_x \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$.
- $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$.
- Entanglement is when $\psi = \sum \alpha_{ij} \, |e_i\rangle \otimes |f_j\rangle \neq c|u\rangle \otimes |v\rangle$ for any $|u\rangle, |v\rangle$: not factorizable.
- Bell states are $|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)$, $|\Psi_i\rangle = (\mathbb{1} \otimes \sigma_i)|\Psi_0\rangle$, orthonormal entangled basis of two 2-state systems.
- Quantum teleportation requires classical bits... it uses entangled Bell states and 3 particles A, B, C.
- No-cloning theorem rests on limits of unitary operators. They can clone dim V orthogonal vectors known in advance.

April 20th (6:00am) Angular Momentum and Central Potentials II

- Dot and cross product rules: be bold, but remember what commutes and fear not to rederive to check!

- $\hat{r} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$, $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$, $\hat{r}\cdot\hat{p} = \hat{p}\cdot\hat{r} + [r_i, p_i] = \hat{p}\cdot\hat{r} + 3i\hbar$,

$$\hat{r}\times\hat{p} = \underbrace{-\hat{p}\times\hat{r}}_{\substack{\text{sign from } \varepsilon_{ijk} \\ \text{switch — always} \\ \text{think, "why?"}}} + \varepsilon_{ijk}[x_j, p_k] = -\hat{p}\times\hat{r} + \underbrace{\varepsilon_{ijk}\delta_{jk}}_{\substack{\text{symmetric and} \\ \text{antisymmetric superpower}}} = -\hat{p}\times\hat{r}.$$

- $\hat{L} = \hat{r}\times\hat{p}$. And $\hat{r}\cdot\hat{L} = \hat{p}\cdot\hat{L} = 0$ by symmetry antisymmetry.
- Critical identity: $\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}$. Useful: $\hat{a}\cdot(\hat{b}\times\hat{c}) = (\hat{a}\times\hat{b})\cdot\hat{c}$.
- Key techniques: caution to not commute operators,
  symmetry - antisymmetry,
  ε_{ijk} identity.

- Exercise: show $\hat{r}$ and $\hat{p}$ satisfy $[\hat{L}_i, \hat{a}_j] = i\hbar\varepsilon_{ijk}\hat{a}_k$.
- Intuition: vector under rotations feel the action of $\hat{L}_i$ in commutators, while scalars under rotation don't: $[\hat{L}_i, \hat{u}_j] = i\hbar\varepsilon_{ijk}\hat{u}_k$ vs. $[\hat{L}_i, \hat{z}] = 0$.
- Intuition: for $\hat{u}, \hat{v}$ vectors under rotation,
  $\hat{u}\cdot\hat{v}$ is scalar, $\hat{u}\times\hat{v}$ is vector (just like classically!)

- Key consequences: $[\hat{L}_i, \hat{L}_j] = [\hat{L}_i, (\hat{r}\times\hat{p})_j] = i\hbar\varepsilon_{ijk}(\hat{r}\times\hat{p})_k = i\hbar\varepsilon_{ijk}\hat{L}_k$,
  $[\hat{L}_i, \hat{r}\cdot\hat{p}] = [\hat{L}_i, \hat{r}^2] = [\hat{L}_i, \hat{p}^2] = [\hat{L}_i, \hat{L}^2] = 0$

  because all of $\hat{r}, \hat{p}, \hat{L}$ are vectors under rotation.

- Define $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$. Then $[\hat{J}_+, \hat{J}_-] = i\hbar \hat{J}_z$, $(\hat{J}_z, \hat{J}_\pm) = \pm\hbar \hat{J}_\pm$, $\hat{J}^2 = \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z$, and $\hat{J}_\pm |j,m\rangle = \sqrt{j(j+1) - m(m\pm1)}\; |j,m\pm1\rangle$.
- Through all your adventures in angular momentum, remember the key intuitions of where each tool comes from.
- For a central potential, $[\hat{L}_i, \hat{H}] = 0$ (because $[\hat{L}_i, \underbrace{\hat{p}\cdot\hat{p}}_{\text{vector}\cdot\text{vector} = \text{scalar}}] = 0$, $[L_i, f(\hat{r}^2)] = 0$)
- All the familiar facts about $|n,\ell,m\rangle$, $2\ell+1$, no degeneracies.
- Rayleigh formula expresses plane waves as spherical harmonics.
- 3D isotropic oscillator is $\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{r}^2$, hence angular multiplets.
- Multiplets are $\ell=0$ singlet $|0\rangle$, $\ell=1$ triplet $\hat{a}_x^\dagger|0\rangle, \hat{a}_y^\dagger|0\rangle, \hat{a}_z^\dagger|0\rangle$, then prelude to addition of angular momentum with $\ell=2$ having $0 \oplus 2$. At fixed E, no two $\ell' = \ell$ multiplets because that would mean a degeneracy in the 1D radial potential.

April 21^{st} ($6{:}00^{am}$) Addition of Angular Momentum

- Addition of angular momentum is NOT a mystery. Each space $\mathcal{H}_1 \otimes \mathcal{H}_2$ has a normal J_i algebra. Then $J_i = J_i^{(1)} + J_i^{(2)} = J_i^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes J_i^{(2)}$. The second form is key for intuition! For calculations $J_\pm^{tot} = J_x^{tot} \pm i J_y^{tot}$ is great.
- Always check normalization!
- Superpower: $\hat{S}_e \cdot \hat{S}_p = \frac{1}{2}(\hat{S}^2 - \hat{S}_e^2 - \hat{S}_p^2)$, and for spin $\hat{S}_e^2 = \hat{S}_p^2 = \frac{3\hbar^2}{4}$.
- Intuition: coupled basis diagonalizes $L \cdot S$ type perturbations, because $L \cdot S \sim J^2 - \text{const}\ \mathbb{1}$.
- Spin-orbit coupling $L \cdot S$ (relativistic electron feels proton magnetic field) again leverages $L \cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$.
- To calculate corrections, select a fixed ℓ so $L^2 = \text{const}$.

$\longrightarrow$ Question: why can we select $\ell = 1$? Doesn't $\ell = 0$ in $n = 2$ have the same energy, so shouldn't we include it in our degenerate subspace calculation?

- Clebsch – Gordon rule: $\langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j m \rangle = 0$ unless $m_1 + m_2 = m$.
- Allowed j values: $j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|$.

April 22nd (6:00am) Charged Particles in Electromagnetic Fields

- Potentials are the fundamental object:

$$\vec{B} = \vec{\nabla} \times \vec{A} \text{ (locally)}, \quad \vec{E} = -\nabla \Phi - \frac{1}{c}\frac{\partial A}{\partial t} \text{ (locally)}.$$

Physics is invariant under a gauge transformation

$$A' = A + \nabla \Lambda, \quad \Phi' = \Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \quad \psi' = \exp\left(i \frac{q\Lambda}{\hbar c}\right)\psi.$$

Units: $[A] = T \cdot m$, $[\Phi] = T \cdot m$.

- Minimal coupling $\hat{p} \to \hat{p} - \frac{q}{c}\vec{A}(\hat{x},t)$.

Units: $q \cdot T =$ force from Lorentz law $(F = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B})$, so $qT\cdot s =$ momentum.

- Hamiltonian becomes $\hat{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\hat{x},t)\right)^2 + q\,\Phi(\hat{x},t)$.
- Coulomb gauge is any with $\vec{\nabla}\cdot\vec{A} = 0$. It simplifies $\hat{H}$.
- Flux quantum $\hat{\Phi}_0 = \frac{2\pi\hbar c}{q}$ is minimal magnetic flux out of a torus.
- Landau levels for particle charge q, mass m in $\vec{B} = B_0\hat{z}$ form harmonic oscillator, $\Delta E = \hbar\omega_c$, $\omega_c = \frac{qB}{mc} = \frac{v}{r} \Leftarrow q\frac{v}{c}B = m\frac{v^2}{r}$.

The length scale $\ell_B^2 = \frac{\hbar}{m\omega} = \frac{\hbar c}{qB}$.

- Superpower: when doing Landau level problems, keep track of units to catch silly math errors. Also, no silly mistakes!
- Infinite degeneracies in k_x for n^{th} level of the oscillator:

$\psi(x,y) = \varphi_n(y-y_0)\, e^{ik_x x}$ with $y_0 = -k_x \ell_B^2$, φ_n the n^{th} oscillator state.

- Finite degeneracies $\frac{1}{2\pi}\frac{L_x L_y}{\ell_0^2} = \frac{\text{flux}}{\text{flux quantum}} = \frac{\Phi}{\Phi_0}$ if restricted to L_x by L_y square: intuition is every state occupies $2\pi\ell_0^2$ space.

April 23rd (7:00 am) Problem Solving Jam!

24.1) DONE!

24.6) DONE! (30 minutes, and that was a midterm problem!)

April 24th (6:00 am) Time-Independent Perturbation Theory

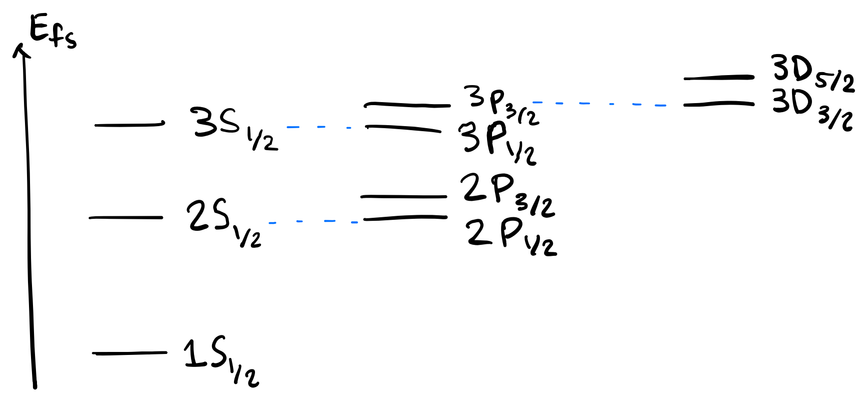
- Nondegenerate: derivation is by postulating power series solution and plugging into Schrödinger.
- Useful choice: assume $\langle n^{(0)} | n^{(k)} \rangle = 0$ if $k \neq 0$ (but not $\langle n^{(k)} | n^{(k)} \rangle = 1$)
- $E_n^{(k)} = \langle n^{(0)} | \delta H | n^{(k-1)} \rangle$. Useful for some calculations.
- $|n^{(1)}\rangle = -\sum_{k \neq n} \frac{\delta H_{kn}}{E_k^{(0)} - E_n^{(0)}} |k^{(0)}\rangle$, $E_n^{(2)} = -\sum_{k \neq n} \frac{|\delta H_{kn}|^2}{E_k^{(0)} - E_n^{(0)}}$.
- Intuition: level repulsion (higher pushes down). Put $E_k^{(0)} - E_n^{(0)}$ so that ground state is pushed down ($E_k^{(0)} - E_0^{(0)} > 0$, so $-$ out front).

Strategy to calculate is always to PAUSE a moment to think, "What is the best way to calculate δH_{kn}?"

- Use intuition like, a quartic potential on an oscillator will increase the energy because the potential is going up.
- Degenerate perturbations: for $[\delta H]$ the perturbation restricted to the degenerate subspace, the good basis diagonalizes $[\delta H]$. $E_{nI}^{(1)} = [\delta H]_{II}$, the I^{th} eigenvalue.
- Useful lemma: $[\delta H]_{kl} = 0$ if $\exists\ [\hat{H}, \hat{K}] = 0$ s.t. $|\psi_k^{(0)}\rangle, |\psi_l^{(0)}\rangle$ have different $\hat{K}$ eigenvalues. The basis is good if $\hat{K}$ is nondegenerate.
- For degenerate state corrections and 2nd order energy, key is splitting up contributions from in and out of degenerate subspace: V_n and $V_\perp$.
- $E_{nI}^{(2)} = -\sum_p \frac{|\delta H_{pI}|^2}{E_p^{(0)} - E_n^{(0)}}$ matches nondegenerate second order, only summed over states p outside the degenerate subspace.

April 25^{th} ($6{:}00^{am}$) Time-Independent Perturbation Theory

- Intuition: coupled basis matters because $\hat{J}$ is conserved in the relativistic correction.
- Label for coupled state multiplets is nL_j, e.g. $3P_{3/2}$.
- All fine structure of order $\alpha^4 mc^2$.
- When stuck in a calculation,
  1. Check units.
  2. Think of what commutes.
  3. Write down the Schrödinger equation.
- Relativistic correction uses uncoupled as good basis. Since the answer only depends on n, ℓ, it remains good in coupled.
- Fine structure adds j dependence, but no ℓ or m_j. Higher j means higher energy. All states shifted down compared to original.



- Weak field Zeeman splits by m_j. Technique: degeneracies were in ℓ, m_j, so showing $\hat{L}^2, \hat{J}_z$ commute is enough for good basis in the degeneracy.
- Useful trick: $\hat{L}\cdot\hat{S} = \hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+ + 2\hat{L}_z\hat{S}_z$

April 26^{th} ($6:00^{am}$) Problem Solving Jam!

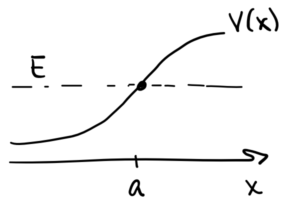
25.7) DONE! [Lesson: good basis in degenerate perturbation theory.]

25.9) SOLVED ENOUGH TO LEARN! [Lesson: unbroken degeneracy is hard.]

25.10) DONE! [Lesson: exp. value of scalar operator is invariant under m.]

25.15) DONE! [Lesson: $\frac{e^2}{r} = \frac{e^2}{\sqrt{\hat{r}^2}}$ is a scalar under rotations. And $\hat{r} \neq r$.]

April 27th (6:00am) WKB and Semiclassical Approximation

- Local momentum $p(x) = 2m(E-V(x))$, local wavelength $\lambda(x) = \frac{2\pi\hbar}{p(x)}$.
- With $\Psi = \sqrt{\rho}\exp(\frac{i}{\hbar}S)$, probability current is $J = \rho\frac{\nabla S}{m}$.
- WKB hypothesis is $\Psi = \exp(\frac{i}{\hbar}S)$, $S(x) \in \mathbb{C}$, plug into Schrödinger, power expand in $\hbar$. Result: $\Psi(x) = \frac{A}{\sqrt{p(x)}}\exp\left(\frac{i}{\hbar}\int_{x_0}^{x} dx' p(x')\right)$.
- Classically allowed: $\Psi(x) = \frac{A}{\sqrt{k(x)}}\exp(i\int\ldots) + \frac{B}{\sqrt{k(x)}}\exp(-i\int\ldots)$.
- Classically forbidden: $\Psi(x) = \frac{C}{\sqrt{\kappa(x)}}\exp(-\int\ldots) + \frac{D}{\sqrt{\kappa(x)}}\exp(\int\ldots)$.
- WKB valid when $\left|\frac{d\lambda}{dx}\right| \ll 1$.
- 

$$\frac{2}{\sqrt{k(x)}}\cos\left(\int_x^a k(x')dx' - \frac{\pi}{4}\right) \Longleftarrow \frac{1}{\sqrt{\kappa(x)}}\exp\left(-\int_a^x \kappa(x')dx'\right)$$

$$\frac{-1}{\sqrt{k(x)}}\sin\left(\int_x^a k(x')dx' - \frac{\pi}{4}\right) \Rightarrow \frac{1}{\sqrt{\kappa(x)}}\exp\left(\int_a^x \kappa(x')dx'\right)$$

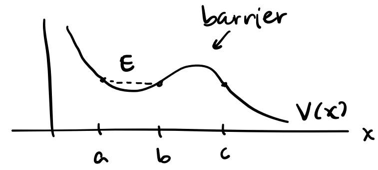
- Connection formulae go in one direction. Different for $V'>0$ vs. $V'<0$.
- Quantization: $\int_a^b k(x')dx' = (n+\frac{1}{2})\pi$ for two turning points

  Derivation is simply connection formulae. Everything in WKB makes sense with whole approximation apparatus in mind.
- Intuition: you can connect away from a decaying exponential or into a growing exponential.
- Tunneling strategy: start $x \gg b$ with transmitted wave, then connect across barrier, keep only the good decaying exponential, and connect again into the incident wave ($\cos \sim$ incident + reflected).
- Result: T_{WKB} = transmission probability $= \exp(-2\int_a^b k(x')dx')$.
- Lifetime τ of decay means $P(t) = e^{-t/\tau}$. Compute by letting particle

bounce in well until it tunnels out.

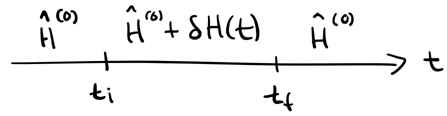
Use semiclassical velocity $v(x) = \frac{p(x)}{m}$

to estimate time between bounces.



April 28th (6:00am) Time-Dependent Perturbation Theory

- Setup is a small $\delta H(t)$, on then off:



- Interaction picture $|\tilde{\Psi}(t)\rangle = e^{i\hat{H}^{(0)}t/\hbar}|\Psi(t)\rangle$ quiesces the action of $\hat{H}^{(0)}$.
  Key to remember all results are in this picture when calculating, e.g. $\widetilde{\delta H} = e^{i\hat{H}^{(0)}t/\hbar}\,\delta H\, e^{-i\hat{H}^{(0)}t/\hbar}$. Then $i\hbar \frac{d|\tilde{\Psi}(t)\rangle}{dt} = \widetilde{\delta H}\,|\tilde{\Psi}(t)\rangle$.

- Writing in a basis and plugging into the Schrödinger equation gives
  $i\hbar \dot{c}_m(t) = \sum_n e^{i\omega_{mn}t}\,\delta H_{mn}(t)\,c_n(t)$, $\omega_{mn} = \frac{1}{\hbar}(E_m - E_n)$, $|\tilde{\Psi}(t)\rangle = \sum_n c_n(t)|n\rangle$.
  General strategy: write down these coupled diff. eqs.
  Perturbative expansion just simplifies.

- $|\tilde{\Psi}(t)\rangle = |\tilde{\Psi}^{(0)}(t)\rangle + \lambda|\tilde{\Psi}^{(1)}(t)\rangle + \lambda^2|\tilde{\Psi}^{(2)}(t)\rangle + O(\lambda^3)$.
  Initial conditions $|\tilde{\Psi}^{(n)}(0)\rangle = 0$ for $n \geq 1$, $|\tilde{\Psi}^{(0)}(t)\rangle = |\Psi(0)\rangle$.
  Schrödinger $i\hbar\partial_t|\tilde{\Psi}(t)\rangle = \lambda\widetilde{\delta H}|\tilde{\Psi}(t)\rangle$ gives $i\hbar\partial_t|\tilde{\Psi}^{(n+1)}(t)\rangle = \widetilde{\delta H}\,|\tilde{\Psi}^{(n)}(t)\rangle$.
  Successive substitutions yield, e.g., $|\tilde{\Psi}^{(2)}(t)\rangle = \int_0^t dt' \frac{\widetilde{\delta H}(t')}{i\hbar}\int_0^{t'} dt'' \frac{\widetilde{\delta H}(t'')}{i\hbar}|\Psi(0)\rangle$.

- For $n \neq m$, keeping only $O(\lambda)$ term gives
  $$P_{n\leftarrow m}(t) = |\langle n|\tilde{\Psi}(t)\rangle|^2 = \left|\int_0^t dt' \frac{\langle n|\widetilde{\delta H}|m\rangle}{i\hbar}\right|^2 = \frac{1}{\hbar^2}\left|\int_0^t dt'\, e^{i\omega_{nm}t}\,\delta H_{nm}(t)\right|^2.$$
  These results are all derived. You can really know them, recreate them!

- In basis vectors, $|\tilde{\Psi}^{(1)}(t)\rangle = \sum_n c_n^{(1)}(t)|n\rangle = \int_0^t dt' \frac{\widetilde{\delta H}(t')}{i\hbar}\overbrace{\sum_n c_n(0)|n\rangle}^{|\Psi(0)\rangle}$.
  Exercise: rederive $c_m^{(1)}(t)$ from the above.

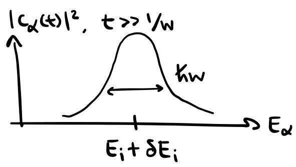
- Special cases: Constant $\delta H \rightarrow P_{f\leftarrow i}(t_0) = \frac{|\delta H_{fi}|^2}{\hbar^2}F(\omega_{fi}, t_0)$, $F(\omega, t) = \frac{\sin^2(\frac{\omega t}{2})}{(\frac{\omega}{2})^2}$.

  Harmonic $\delta H = 2H'\cos(\omega t) \rightarrow \cos$ splits to cause $e^{i(\omega_{fi}+\omega)t_0}$, $e^{i(\omega_{fi}-\omega)t_0}$ in $c_f^{(1)}(t_0)$ which when integrated create denominators that make either $\omega \approx \omega_{fi}$ or $\omega \approx -\omega_{fi}$ dominate as absorption/stimulated emission. Math works!

- Conditions for validity: $t_0 \gg \frac{1}{\omega}$ so δH is identifiable as oscillatory
  $t_0 \ll \frac{\hbar}{|H'_{fi}|}$ so that resonant probability $\frac{|H'_{fi}|}{\hbar^2}t_0^2 \ll 1$

April 29^{th} (7:00 am) Time-Dependent Perturbation Theory

- Discrete $\to$ continuum transition uses box construct.
- Count density of states $\rho(E)$ by $L^{-3/2}e^{i\vec{k}\cdot\vec{x}} \Rightarrow k_i L = 2\pi n_i$, $\Delta N = dn_x dn_y dn_z$
  $\Rightarrow \Delta N = \left(\frac{L}{2\pi}\right)^3 d^3k$, $E = \frac{\hbar^2 k^2}{2m} \Rightarrow k\,dk = \frac{m}{\hbar^2}dE$, $d^3k = k^2 dk\, d\Omega \Rightarrow \Delta N = \underbrace{\left[\left(\frac{L}{2\pi}\right)^3 \frac{m}{\hbar^2} k\, d\Omega\right]}_{\rho(E)} dE$.
- First trick is $\sum_{\text{states}} \cdots \to \int \cdots \rho(E)\, dE$.
- Second trick is assume $|V_{fi}|^2 \rho(E) \sim$ constant because $F(\omega_{fi}, t_0)$ suppresses contributions aside from $E_f \sim E_i$ where it is large.
- Fermi's Golden Rule: transition rate $w = \frac{2\pi}{\hbar}|V_{fi}|^2 \rho(E_f)$, $E_f = E_i$ for constant, or $E_f = E_i \pm \hbar\omega$ for harmonic.
- Validity requires $t_0 w \ll 1$ and $t_0 \gg \frac{\hbar}{\Delta_E(K)}$, $\Delta_E(K)$ = energy for $K(E) = |V_{fi}|^2\rho(E)$ to change comparably to K. Simplest condition is $\left|\frac{d\hbar w}{dE}\right|_{E_f} \ll 1$.
- Decay of discrete to continuum:



- Ionization of hydrogen: demand $\underbrace{Ry \ll \hbar\omega}_{\text{so ejected electron is momentum eigenstate}} \ll 1722\, Ry$, with $\hbar\omega \ll 1722\, Ry$ so magnetic field constant over the atom

  Electric dipole approximation is $\delta H = 2\left(\frac{-e}{m\omega}\vec{E}_0 \cdot \hat{p}\right)\cos(\omega t)$.
  It comes from $\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$ with $\vec{\nabla}\cdot\vec{A} = 0$ gauge and $\vec{A}\cdot\vec{A} \sim |E|^2$ ignored.

- Ionization $\frac{dw}{d\Omega} = \frac{64}{\pi}\frac{ma_0^2}{\hbar^2}\frac{(eE_0a_0)^2}{\hbar}\frac{1}{(k_e a_0)^9}\cos^2\theta$ (for $\vec{E}$ in $\hat{z}$ direction, $\theta \to \hat{z}$)
  $\hbar k_e$ = momentum of ejected electron

April 30th (6:00am) Time-Dependent Perturbation Theory

- $U(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3 \, d\omega}{e^{\beta\hbar\omega} - 1}$ is energy per unit volume in frequency range $d\omega$ for thermal blackbody radiation.

- Einstein's ingenious argument $\Rightarrow$ $\underbrace{B_{ab}}_{\text{stimulated absorption rate}} = \underbrace{B_{ba}}_{\text{stimulated emission rate}}$, $\frac{\overbrace{A}^{\text{spontaneous emission rate}}}{B_{ab}} = \frac{\hbar\omega_{ba}^3}{\pi^2 c^3}$.

- Intuition: stimulated dominates for high temperature, spontaneous dominates for low temperature.

- Intuition: to think like an approximator, write down the whole ugly beast and ask what can go away.

  Example: $\vec{E}, \vec{B} \to$ just $\vec{E}$ for nonrelativistic electron

  $\vec{E}(\vec{x}, t) \to$ just $\vec{E}(t)$ for $\lambda \gg a_0$ for visible light

- $B_{ab} = B_{ba} = \frac{4\pi^2}{3\hbar^2} |\vec{d}_{ab}|^2 U(\omega_{ba})$, $A = \frac{4}{3} \frac{\omega_{ba}^3}{\hbar c^3} |\vec{d}_{ab}|^2$, $(\vec{d}_{ab})_i = q \langle a | x_i | b \rangle$

- Selection symmetry! $\langle n \ell m | \hat{r} | n' \ell' m' \rangle = 0$ unless $\Delta \ell = \pm 1$, $\Delta m = 0, \pm 1$.

  $\Delta m = 0 \Rightarrow \langle a | \hat{x} | b \rangle = \langle a | \hat{y} | b \rangle = 0$

  $\Delta m = 1 \Rightarrow \langle a | \hat{z} | b \rangle = 0$, $\langle a | \hat{x} | b \rangle = \pm i \langle a | \hat{y} | b \rangle$ (from $\langle a | [\hat{L}_z, \hat{x}] | b \rangle$)

May 1st (6:00am) Problem Solving Jam!

- 27.4: DONE! [Lesson: Fourier of Gaussian is $\int_{-\infty}^{\infty} e^{ikx} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}\, e^{-k^2/4a}$]
- 27.6: → CONFUSED → Are we operating in 1D or 3D (for E&M wave)
- 27.7: DONE! (Lesson: When you see "harmonic oscillator," whip out $\hat{a}, \hat{a}^{+}$!]

Adiabatic Approximation

- Adiabatic means slow change to parameter λ of $\hat{H}$:

  $\tau \left|\frac{d\lambda}{dt}\right| \ll |\lambda|$ for τ a natural timescale

- Example natural timescales: $\frac{2\pi}{\omega}$ (period of oscillator) or $\frac{\hbar}{E_1 - E_2}$ (2-state)
- Adiabatic invariant means $|I(t) - I(0)| \to 0$ as $T \to \infty$ for any t, where $\hat{H}$ changes by $\Delta\lambda$ over time $[0,T]$.
- Key intuition: quantum number (i.e. state $|n\rangle$) is an adiabatic invariant.
- Instantaneous eigenstate $|\psi(t)\rangle$ to solve $H(t)|\psi(t)\rangle = E(t)|\psi(t)\rangle$ is a glued-together amalgam of individually solved $H(t_0)|\psi(t_0)\rangle = E(t_0)|\psi(t_0)\rangle$.
- Approximate solution $|\Psi(t)\rangle$ to Schrödinger given instantaneous $|\psi(t)\rangle$:

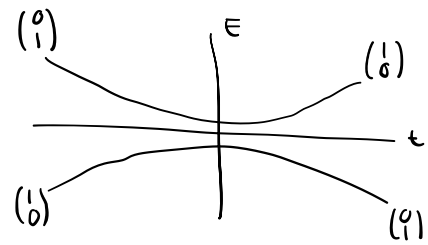
$$|\Psi(t)\rangle \approx c(0)\, \exp\Big(i \underbrace{\int_0^t}_{\gamma(t) = \int_0^t \nu(t')\,dt'} \underbrace{i\langle \psi(t')|\dot{\psi}(t')\rangle}_{\nu(t)} dt'\Big)\, \exp\Big(\underbrace{\tfrac{1}{i\hbar}\int_0^t E(t')\,dt'}_{\theta(t) = \frac{-1}{\hbar}\int_0^t E(t')\,dt'}\Big)\, |\psi(t)\rangle \; .$$

$$= c(0)\, e^{i\gamma(t)} e^{i\theta(t)} |\psi(t)\rangle$$

- Adiabatic theorem: if $|\Psi(0)\rangle = |\psi_n(0)\rangle$ is fully an instantaneous eigenstate, then $|\Psi(t)\rangle$ will remain that instaneous eigenstate.

May 2nd (6:00am) Adiabatic Approximation

- Proof of adiabatic theorem uses trick: we can always assume $\langle \psi_k(t) | \dot{\psi}_k(t) \rangle = 0 \ \forall k$ (under replacement $|\psi(t)\rangle \to e^{i\gamma(t)} |\psi(t)\rangle$).
- "Adiabatic transition" means following $|\psi_k(t)\rangle$, even as physical properties of that conserved state may change.
- Landau-Zener 2-state system $\begin{bmatrix} \alpha t/2 & H_{12} \\ H_{12}^* & -\alpha t/2 \end{bmatrix}$ is adiabatic if quantum timescale $T_{12} = \frac{2\pi}{\omega_{12}}$, $\omega_{12} = \frac{|H_{12}|}{\hbar}$, is much less than transition duration $\tau = \frac{|H_{12}|}{\alpha}$. Good exercise to rederive these timescales.



- Berry's phase $\gamma_n(t) = \int_\Gamma \langle \psi_n(\vec{R}) | \nabla_{\vec{R}} | \psi_n(\vec{R}) \rangle \, d\vec{R}$ is geometric, depending only on the path Γ. Call the integrand the Berry connection $A_n(\vec{R})$.
- $\gamma_n(\Gamma)$ is gauge invariant (i.e. under $|\psi_n(t)\rangle \to e^{i\beta(t)} |\psi_n(t)\rangle$) if Γ is a loop.
- Example: $\vec{B}$ field traces loop in (B_0, θ, ϕ) space $\to \gamma_n(t) = -\frac{1}{2}\Omega$ (solid angle traced)
- Born–Oppenheimer approximation uses Berry's phase. Long derivation. The upshot is approximations make molecular dynamics tractable.

May 3^{rd} ($6:00^{am}$) Scattering in One Dimension

- Scattering key idea: superimpose incoming and outgoing wavefunction $e^{-ikx} + e^{ikx+2i\delta}$ with energy-dependent phase shift δ, then call the scattered wave whatever wouldn't have featured in the zero potential case.
- Time delay is if the particle is repulsed or temporarily trapped in the potential. Time advance means the particle is attracted and reflects faster. All scattering times are compared to no potential.
- Stationary phase principle $\Rightarrow$ time delay is $x = v_0 \left(t - 2\hbar \frac{d\delta}{dE}\Big|_{E(k_0)} \right)$ for wavepacket narrowly peaked around k_0.

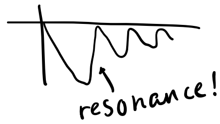
  For emphasis, $\Delta t = 2\hbar \delta'(E)$.

- Another form is $\frac{1}{R} \frac{d\delta}{dk} = \frac{\text{time delay}}{\text{free transit time across range length } R}$.

May 4th (6:00am) Scattering in One Dimension

- Levinson's Theorem: number of bound states $= \frac{1}{\pi}(\delta(E=0) - \delta(E=\infty))$.

- Resonance occurs when the particle gets trapped or localized in the short range potential like a bound state for a long time. Resonance means a peak in $\delta(E)$ followed by rapid growth that crosses $\pi/2$ (mod π).



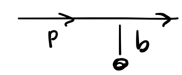
- Resonance condition: $\tan(\delta(k)) = -i$. Then $\text{Re}(k)$ tells the energy at resonance, and lifetime $\tau = \frac{\Delta t}{4} = \frac{ma^2}{2\text{Re}(k)\text{Im}(k)\hbar}$.

- There's a lot to understand about 1D potentials and how they connect to the complex plane.

May 5th (6:00 am)

Scattering in Three Dimensions

- Setup is immovable potential (CoM frame, e.g.), e^{ikz} incident wavefunction. Unitless because e^{ikz} is not normalizable. Range of $V(r)$ is a.
- Energy eigenstate has asymptotic scattering piece $\psi_s(r) = f_k(\theta,\phi)\frac{e^{ikr}}{r}$, spherically symmetric that satisfies $(\nabla^2+k^2)\frac{e^{ikr}}{r} = 0$ for $r > a$. $f_k(\theta,\phi)$ has units of length and is called "scattering amplitude."
- $d\sigma$ = area that, removed from the incident beam, would remove all particles scattered into $d\Omega$ around (θ,ϕ)
- $\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2$ is the key observable result. Also $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$.
- Partial wave method: decompose e^{ikz} into all modes of angular momentum ℓ, each an outgoing and incoming with a phase shift δ_ℓ like in the 1D case. The phase shift is the *maximal variability possible* as $r \to \infty$ that fits with conservation of probability.
- Setting the phase shift ansatz above equal to $\psi(r) = e^{ikz} + f_k(\theta,\phi)\frac{e^{ikr}}{r}$ with e^{ikz} decomposed with no phase shifts, we get f_k in terms of δ_ℓ as $f_k(\theta,\phi) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\, Y_{\ell,0}(\theta)\, e^{i\delta_\ell}\sin(\delta_\ell)$.
- Total cross section $\sigma = \sum_{\ell=0}^{\infty}\sigma_\ell$, with partial-wave cross sections $\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell$. Partial waves don't interfere after integration over $d\Omega$ because orthogonal $Y_{\ell m}$'s cancel.
- Forward direction is $\theta = 0$. Optical theorem says $\sigma = \frac{4\pi}{k}\operatorname{Im}(f_k(\theta=0))$. Result follows from $Y_{\ell 0} \sim P_\ell(\cos\theta)$ with P_ℓ Legendre polynomials.

- Summary intuition: e^{ikz} breaks into partial waves, outgoing and ingoing. Probability conservation tells us these differ by at most a phase shift. We postulate $\psi(r) \to e^{ikz} + f_k(\theta,\phi)\frac{e^{ikr}}{r}$ as $r\to\infty$, which matched up gives $f_k(\theta,\phi)$ in terms of the phase shifts.
- $u_{\ell=0}(r) = C\sin(kr+\delta_0)$ for $r>a$ always, from 1D radial intuition.
- Hard sphere potential needs $u_{\ell=0}(r=a)=0 \implies \delta_0 = -ka$, $\sigma_0 = 4\pi a^2$.
- General computation of δ_ℓ: find radial solution $u_\ell(r)$ for $r<a$, match it to $r(A_\ell j_\ell(kr) + B_\ell n_\ell(kr))$ at $r=a$ threshold, $\tan(\delta_\ell) = \frac{-B_\ell}{A_\ell}$.
- Semiclassically, we expect most scattering for $\ell \lesssim ka$. The classical impact parameter $b=\frac{\ell}{k}$ should not exceed potential range: 

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The centrifugal barrier prevents the particle from reaching the potential.
- Integral equation for scattering: $\psi(r) = \psi_0(r) + \int d^3r'\, G(r-r')U(r')\psi(r')$, where $\psi_0(r)$ is a homogeneous solution of $(\nabla^2+k^2)\psi_0(r)=0$, and where $(\nabla^2+k^2)G(r-r') = \delta^3(r-r')$ is a Green's function of ∇^2+k^2.
Fully, $\psi(r) = e^{i\vec{k}_i\cdot\vec{r}} + \int d^3r\, G_+(r-r')U(r')\psi(r')$, $G_+(r) = \frac{-1}{4\pi}\frac{e^{ikr}}{r}$.
- Born approximation sticks $\psi(r)$ into its own formula (above) to create a power series in the potential $U(r) = \frac{2m}{\hbar^2}V(r)$.
- First Born approximation $\psi(\vec{r}) \approx e^{i\vec{k}_i\cdot\vec{r}} + \int d^3\vec{r}'\, G_+(\vec{r}-\vec{r}')U(\vec{r}')e^{i\vec{k}_i\cdot\vec{r}}$,
$f_k^B(\theta,\phi) = \frac{-1}{4\pi}\int d^3\vec{r}\, e^{i\vec{K}\cdot\vec{r}}U(\vec{r})$, $\vec{K} \equiv k(\vec{n}-\vec{n}_i)$, $\vec{n} = (\theta,\phi)$, $\vec{n}_i$ = incident direction.
- For a central potential, $f_k^B(\theta) = \frac{-2m}{\hbar^2 K}\int_0^\infty dr\, rV(r)\sin(Kr)$, $K = 2k\sin\left(\frac{\theta}{2}\right)$.
- Born approximation is valid for small potentials $|V| \ll \frac{\hbar^2}{ma^2}$ or, for high particle energies, when $|V| \ll \frac{\hbar^2}{ma^2}ka$.

May 6^{th} ($6{:}00^{am}$) Problem Solving Jam!

30.4) DONE! (Except part (c). Lesson: use tricks for $\ell=0$ scattering.)

30.6) DONE! (Except part (c). Lesson: use formula for radial Born approx.)

May 7^{th} ($6:00^{am}$) Problem Solving Jam!

28.5) Confused by wording, will seek advice.

27.6) Half done. Lesson: write out a clear recipe for density of states.